End to end QoS (Quality of Service) is crucial in computer networks, but very hard to study as the systems are in general very complex. We suppose that systems can be modeled by multidimensional Markov processes, which could be very hard to analyse if there is no specific solution form. We propose to apply stochastic comparisons of Markov processes in order to solve this problem. We provide new processes, easier to analyze and representing stochastic bounds (upper or lower) for the original model. In this paper, we propose strong and weak bounding processes for a general queueing network model, and discuss their accuracy for QoS constraints.

1. INTRODUCTION

With Internet technologies, a very large number of users from very distant countries can communicate and exchange files with just a click. Many transactions are executed on distributed systems, generating different packet flows transiting from node to node in a network until the destination. For users satisfaction, networks must provide a sufficient QoS level thus a precise dimensioning of the resources. Markov processes are very efficient models for the quantitative analysis. Unfortunately, mathematical analysis of multidimensional Markov processes could be very difficult. We are interested in stationary and transient behaviours of the system. If the underlying model does not let to apply a specific form like product form, matrix-geometric form to compute the probability distribution of interest (transient, stationary) then numerically computation could be difficult or intractable since the state space increases exponentially with the parameter size.

We propose to use a mathematical method based on stochastic comparisons of Markov processes [13]. The key idea of this method is the following: given a large size Markov process, we bound it by other Markov processes easier to analyse and which provide bounds on performance measures. Different solutions are proposed using stochastic comparisons. The bounding process either has a probability distribution with a specific form, or it is defined on a smaller state space [7, 3].

Related work: A stochastic ordering is defined as a relation order between random variables, or stochastic processes [13]. The most known stochastic order is the strong stochastic ordering (\(\preceq_{st}\)), equivalent to a sample path ordering [13]. When the state space is multidimensional, weak stochastic orderings (\(\preceq_{wk}\) and \(\preceq_{wk^*}\)) can also be defined using increasing sets families [13, 11]. The strong ordering yields to comparisons of increasing functionals (the expectations of all increasing functions of the probability distributions) while the weak ordering is equivalent to the tail probability distribution comparisons. For the strong ordering, the coupling method governed by the events is proposed to compare realisations of the process [8, 9, 6]. For the weak ordering the increasing set method with events is applied in order to limit the number of increasing sets [4]. We apply these methods on a general queueing network similar to a Jackson network except that queues have a finite capacity. This system is very general, and thus can represent for example a network of routers with a general topology. As this system is very difficult to study, we propose to define from the original system two bounding systems by imposing independence between queues. The first idea is to make queue capacities infinite to obtain a Jackson network. The second one is to cut the links between the queues to have a system with independent M/M/1/K queues [11, 10]. We prove using the coupling that the Jackson network represents an upper bound for the strong ordering (strong bound).
We apply the increasing set formalism in order to prove that the second system represents an upper weak bound.

To the best of our knowledge, there is no study which aims to compare the quality of weak and strong bounds. In this paper, by considering different input parameter values (routing probabilities, load) we compute blocking probability bounds from weak and strong bounding models, and we compare them. We try to propose the best bounding system with respect to the input parameters. The relevance of this paper is to study the accuracy of the bounds in order to improve the resource dimensioning. Thus in a complex network, given blocking probability requirements, we are able to give the buffer size of any node satisfying these constraints.

This paper is organised as follows: we first present a brief introduction of stochastic comparison method. In section 3, we derive bounds on a general queueing model. In section 4, we compute the blocking probability from bounding models and give numerical results to determine the best bounding system with respect to input parameters. We also study the buffer dimensioning problem by means of bounding threshold blocking probabilities. Finally, we conclude and give comments about further research items.

2. STOCHASTIC COMPARISONS

We suppose that the considered telecommunication system is modelled by a multidimensional Continuous-Time Markov Chain (CTMC) denoted by \{X_1(t), t \geq 0\} taking values on state space \(E\). The considered performance or reliability measure such as loss probability, availability is defined as a functional at time \(t\) over a subset of states \(A \subset E\):

\[
R_1(t) = \sum_{x \in A} \Pi_1(x,t) f(x)
\]

where \(f : E \rightarrow \mathbb{R}^+\) is an increasing reward function and \(\Pi_1(x,t)\) is the probability to be in the state \(x\) at time \(t\). For \(t \rightarrow \infty\), if the process has a stationary behaviour, then we denote by \(\Pi_1(x)\) the stationary probability to be in state \(x\), and \(R_1\) represents the measure of interest computed from the stationary probability distribution \(\Pi_1\). In the cases where the computation of \(\Pi_1(t)\) (or \(\Pi_1\)) is difficult or impossible, we propose to apply the stochastic comparison method, which means to bound (upper or lower) \{\(X_1(t), t \geq 0\)\} by \{\(X_2(t), t \geq 0\)\}, in the sense of the \(\preceq_\phi\)-ordering

\[
\{X_1(t), t \geq 0\} \preceq_\phi \{X_2(t), t \geq 0\}
\]

or

\[
\{X_2(t), t \geq 0\} \preceq_\phi \{X_1(t), t \geq 0\}
\]

such that \(\{X_2(t), t \geq 0\}\) is easier to analyse (see Fournier et al. [7]). The bounding model may have a closed form solution or defined in a reduced state space which makes easier the computation of the bounding distribution \(\Pi_2(t)\). Therefore the bound on the measure of interest \(R_2(t)\) is computed from the bounding distribution \(\Pi_2(t)\), such that

\[
R_1(t) \leq R_2(t) \text{ or } R_2(t) \leq R_1(t)
\]

We now present the main concepts of stochastic comparisons of random variables and Markov chains.

2.1. Stochastic ordering theory

Let \(E\) be a discrete, and countable state space, and \(\preceq\) be at least a preorder (reflexive, transitive but not necessarily an anti-symmetric binary relation) on \(E\). We suppose that \(E\) is a multidimensional state space with discrete components thus it is suitable for the queueing network representation. We consider two random variables \(X\) and \(Y\) defined respectively on \(E\), and their probability measures given respectively by the probability vectors \(p\) and \(q\) where \(p[i] = \text{Prob}(X = i), \forall i \in E\) (resp. \(q[i] = \text{Prob}(Y = i), \forall i \in E\)). The well-known sample path ordering \(\preceq_{st}\) is defined as follows [13]:

**Definition 1** \(X \preceq_{st} Y \Leftrightarrow E[\{f(X)\}] \leq E[\{f(Y)\}] \forall f : E \rightarrow \mathbb{R}^+ \preceq \text{ increasing whenever the expectations exist.}

There exist other orderings implying weaker constraints. In [11], the weak ordering \(\preceq_{wk}\) has been defined, it is equivalent to tail distribution comparisons, and \(\preceq_{wk}\) serves the same role for cumulative distribution functions. In the case of multidimensional state spaces, the increasing set formalism is a general formalism for stochastic comparisons. Let \(\Gamma \subseteq E\), we denote by:

\[
\Gamma \vdash = \{x \in E \mid y \geq x, x \in \Gamma\}
\]

An increasing set \(\Gamma\) is a set such that for any element \(x\), all elements which are greater than \(x\) are also in \(\Gamma\). It is formally defined as follows [11]:

**Definition 2** \(\Gamma \subseteq E\ is\ called\ an\ increasing\ set\ if\ and\ only\ if\ \Gamma = \Gamma \vdash\).

Three stochastic orderings have been defined from families of increasing sets (see Massey [11]). The first one is \(\Phi_{st}(E)\) which is defined from all the
increasing sets of $E$:

$$\Phi_{st}(E) = \{\text{all increasing sets on } E\}. \quad (6)$$

Other families $\Phi_{wk}(E)$ and $\Phi_{wk^*}(E)$ are defined from particular kinds of increasing sets. For $x \in E$, let

$$\{x\}^\uparrow = \{y \in E, \; y \geq x\} \text{ and } \{x\}^\downarrow = \{y \in E, \; y \leq x\}$$

The set of increasing sets are defined as follows:

$$\Phi_{wk}(E) = \{ \{x\}^\uparrow, \; x \in E \} \quad (7)$$

$$\Phi_{wk^*}(E) = \{ E - \{x\}^\downarrow, \; x \in E \} \quad (8)$$

Let us remark here that when the state space is totally ordered, $\Phi_{wk}(E) = \Phi_{wk^*}(E)$. If $\Phi(E)$ represents one of these families, then a stochastic ordering $\preceq \Phi \in \{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}\}$ can be defined as follows $[11]$:

**Definition 3**

$$X \preceq \Phi Y \iff \sum_{x \in \Gamma} p[x] \leq \sum_{x \in \Gamma} q[x], \forall \Gamma \in \Phi(E) \quad (9)$$

The $\preceq \Phi$ ordering between random variables $(X \preceq \Phi Y)$ is equivalent to the comparison of corresponding probability vectors $(p \preceq \Phi q)$. We have the following inclusion relations: $[11]$:

**Property 1**

$$\Phi_{wk}(E) \subset \Phi_{st}(E), \Phi_{wk^*}(E) \subset \Phi_{st}(E)$$

It follows from these inclusion relations that the ordering $\preceq_{st}$ is generated from the largest family of increasing sets $\Phi_{st}(E)$. So $\preceq_{wk}$ and $\preceq_{wk^*}$ stochastic orderings are weaker than the $\preceq_{st}$ ordering. Thus the following implications are easily derived for the comparison of probability vectors $p$ and $q$ (or random variables $X$ and $Y$) $[11]$:

**Property 2**

$$p \preceq_{st} q \Rightarrow p \preceq_{wk} q \text{ and } p \preceq_{wk^*} q \quad (10)$$

As we will see after, this relationship between stochastic orderings could not be generalised to the comparison of Markov chains.

### 2.2. Markov chain comparisons

We focus on the stochastic comparisons of multidimensional Markov chains, especially for CTMCs. Let $\{X_1(t), t \geq 0\}$ (resp. $\{X_2(t), t \geq 0\}$) be a CTMC taking values on $E$, with infinitesimal generator $Q_1$ (resp. $Q_2$). The $\preceq \Phi$ stochastic comparison $(\preceq \Phi \in \{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}\})$ is defined as follows $[13]$:

**Definition 4**

$\{X_1(t), t \geq 0\}$ is said to be less in the sense of the stochastic order $\preceq \Phi$ than $\{X_2(t), t \geq 0\}$ ($(\{X_1(t), t \geq 0\} \preceq \Phi \{X_2(t), t \geq 0\}$), if and only if:

$$X_1(0) \preceq \Phi X_2(0) \Rightarrow X_1(t) \preceq \Phi X_2(t), \forall t > 0$$

The stochastic comparison of Discrete Time Markov Chains (DTMCs) can be similarly defined, by comparing the chains at each step $n$ ($n \in \mathbb{N}$). We can define the $\preceq \Phi$ stochastic comparison of time homogeneous CTMCs using their infinitesimal generators $[11]$:

**Definition 5**

$\{X_1(t), t \geq 0\} \preceq \Phi \{X_2(t), t \geq 0\}$ if and only for all probability vectors in $E$, we have

$$p \preceq \Phi q \Rightarrow p \exp(tQ_1) \preceq \Phi q \exp(tQ_2)$$

Using the probability transition matrices we can define the comparison of DTMCs. Let $\{X_1(n), n \geq 0\}$ (resp. $\{X_2(n), n \geq 0\}$) with probability transition matrix $P_1$ (resp. $P_2$):

**Definition 6**

$\{X_1(n), n \geq 0\} \preceq \Phi \{X_2(n), n \geq 0\}$ if and only for all probability vectors in $E$, we have

$$p \preceq \Phi q \Rightarrow pP_1 \preceq \Phi q P_2$$

The stochastic monotonicity is a property that is often used for stochastic comparisons of Markov chains. It is defined as an increasing (or decreasing) in time of the process $[13]$:

**Definition 7**

$\{X(t), t \geq 0\}$ is said to be $\preceq \Phi$-monotone, if

$$X(t) \preceq \Phi X(t + \tau), \forall t \geq 0, \forall \tau \geq 0. \quad (11)$$

Note that definition 7 can be similarly given for DTMCs. The monotonicity can be also defined through probability transition matrices:

**Definition 8**

$\{X_1(n), n \geq 0\}$ is said to be $\preceq \Phi$-monotone, if: for all probability vectors in $E$:

$$p \preceq \Phi q \Rightarrow pP_1 \preceq \Phi q P_1$$

Note that comparison and monotonicity of CTMCs can be defined using the uniformised Markov chains $[13]$ because they have the same properties. $\{X(t), t \geq 0\}$ is uniformisable if and only if:

$$|Q| < \infty \text{ where } |Q| = \sup_{x \in E} Q(x, x)$$

So we can define the uniformised Markov chain $\{X^n(n), n \geq 0\}$ with stochastic matrix $P^n$ such that:

$$P^n = I + \frac{1}{\lambda} Q \text{ where } \lambda \geq 2 \cdot |Q|$$
The stochastic comparison of CTMCs \( \{X_1(t), t \geq 0\} \) and \( \{X_2(t), t \geq 0\} \) can be defined using definition 5, and also definition 6 through the corresponding uniformized Markov chains \( \{\hat{X}^1_n(n), n \geq 0\} \) and \( \{\hat{X}^2_n(n), n \geq 0\} \). Next, we focus on the methods to establish stochastic comparisons of Markov chains. First, we present the coupling method.

2.3. The coupling method

The coupling method is a well-known method for the comparisons of probability measures and Markov processes (see Lindvall [8]). We present first this method for the \( \preceq_{st} \) stochastic comparisons, and then to establish the \( \preceq_{st} \) monotonicity. The \( \preceq_{st} \) comparison of CTMCs is equivalent to the definition of a coupled version of the processes in order to compare their sample paths. For the coupling of \( \{X_1(t), t \geq 0\} \) and \( \{X_2(t), t \geq 0\} \), we define two CTMCs on \( E, \{\hat{X}_1(t), t \geq 0\} \) and \( \{\hat{X}_2(t), t \geq 0\} \) such that:

- \( \{\hat{X}_1(t), t \geq 0\} \) has the same infinitesimal generator as \( \{X_1(t), t \geq 0\} \)
- \( \{\hat{X}_2(t), t \geq 0\} \) has the same infinitesimal generator as \( \{X_2(t), t \geq 0\} \)

The \( \preceq_{st} \) comparison can be obtained using the coupling of the processes to check if the sample paths are ordered [8]:

**Theorem 1** \( \{X_1(t), t \geq 0\} \) \( \preceq_{st} \) \( \{X_2(t), t \geq 0\} \) if there exists a coupling \( \{(\hat{X}_1(t), \hat{X}_2(t)), t \geq 0\} \), such that \( \forall \omega \in \Omega, \forall t > 0 : \hat{X}_1(0)(\omega) \preceq \hat{X}_2(0)(\omega) \Rightarrow \hat{X}_1(t)(\omega) \preceq \hat{X}_2(t)(\omega) \) (12)

The coupling can be also applied in order to check the monotonicity of a process. In fact, the strong monotonicity is equivalent to the coupling of the process with itself [8]. In order to establish the monotonicity of \( \{X(t), t \geq 0\} \), we define two processes:

- \( \{\hat{X}(t), t \geq 0\} \) and \( \{\hat{X}'(t), t \geq 0\} \)

 governed by the same infinitesimal generator matrix as \( \{X(t), t \geq 0\} \), representing different realisations of \( \{X(t), t \geq 0\} \) with different initial conditions. The theorem of the monotonicity using the coupling is as follows (see Lindvall [8]):

**Theorem 2** \( \{X(t), t \geq 0\} \) is said to be \( \preceq_{st} \) - monotone if and only if there exists the coupling \( \{(\hat{X}(t), \hat{X}'(t)), t \geq 0\} \) such that \( \forall \omega \in \Omega : \hat{X}(0)(\omega) \preceq \hat{X}'(0)(\omega) \Rightarrow \hat{X}(t)(\omega) \preceq \hat{X}'(t)(\omega), \forall t > 0 \)

In the case of Markovian discrete - event models, one can compare the evolutions of the realisations due to events. In the case of multidimensional Markov processes, the coupling is more difficult to apply (see [8]) since the system is naturally endowed by a partial order, and governed by more events.

2.4. Increasing set method

In this subsection, we give conditions for comparability and monotonicity of Markov processes by means of increasing sets. We highlight those that can be applied only for the \( \preceq_{st} \) ordering, and those that can be applied for the three orderings : \( \preceq_{st}, \preceq_{wk} \) and \( \preceq_{wk^*} \). The stochastic comparisons can be verified using the probability transition matrices for DTMCs. In the case of the \( \preceq_{st} \) ordering, it can be verified by comparing the rows of the matrices for comparable state \( x \) and \( y \) such that \( x \preceq y \) [11, 13]:

**Theorem 3** \( \{X_1(n), n \geq 0\} \) \( \preceq_{st} \) \( \{X_2(n), n \geq 0\} \) if and only if

\[
\forall \Gamma \in \Phi_{st}(E), \forall x \preceq y \sum_{z \in \Gamma} P_1(x, z) \leq \sum_{z \in \Gamma} P_2(y, z)
\]

In the case of CTMCs, we have also the theorem using infinitesimal generators [11, 13]:

**Theorem 4** \( \{X_1(t), t \geq 0\} \) \( \preceq_{st} \) \( \{X_2(t), t \geq 0\} \) if and only if

\[
\forall \Gamma \in \Phi_{st}(E), \forall x \preceq y \mid x, y \in \Gamma \text{ or } x, y \notin \Gamma \sum_{z \in \Gamma} Q_1(x, z) \leq \sum_{z \in \Gamma} Q_2(y, z)
\]

Note that in theorem 4, we have to consider the states \( x \preceq y \) such that either they are both in the increasing sets or not, because of the negative term in the diagonal. These theorems could not be generalized to the other weaker orderings \( \preceq_{wk} \) and \( \preceq_{wk^*} \). In the general case of the \( \preceq_{wk} \)-comparisons, the comparison of CTMC is also established using inequality conditions on the generators [11]. We have the following theorem (see Massey [11]):

**Theorem 5** \( \{X_1(t), t \geq 0\} \) \( \preceq_{wk} \) \( \{X_2(t), t \geq 0\} \), if the following conditions are satisfied:

1. \( X_1(0) \preceq_{wk} X_2(0) \)
2. \( \{X_1(t), t \geq 0\} \) or \( \{X_2(t), t \geq 0\} \) is \( \preceq_{wk} \)-monotone
3. Comparison of infinitesimal generators \( Q_1, Q_2 \):

\[
\forall \Gamma \in \Phi(E), \forall x \in E, \sum_{z \in \Gamma} Q_1(x, z) \leq \sum_{z \in \Gamma} Q_2(x, z).
\]
It follows from Theorem 5 that the $\preceq_{w}$-monotonicity of one of the processes is a sufficient condition (condition (2)) to establish stochastic comparisons. For the $\preceq_{st}$ ordering, the stochastic monotonicity of DTMC can be obtained from the following theorem:

**Theorem 6** \{X_{1}(n), n \geq 0\} is $\preceq_{st}$-monotone if and only if, \(\forall \Gamma \in \Phi_{st}(E)\)
\[
\forall x \preceq y \sum_{z \in \Gamma} P_{1}(x, z) \leq \sum_{z \in \Gamma} P_{1}(y, z)
\]

Unfortunately, this property could not be generalised to the $\preceq_{w}$-monotonicity. In [11], it is shown that $\preceq_{w}$-monotonicity of a process is equivalent to the $\preceq_{w}$-monotonicity of the generator which is expressed in terms of specific operators.

**2.5. Choice of stochastic comparison method**

The coupling is an intuitive method but applicable only for the $\preceq_{st}$ ordering. The increasing set method can be applied for all $\preceq_{w}$ but it is rather difficult for the $\preceq_{st}$ ordering since one must take into account all the increasing sets. In order to be more precise on how to use these methods, we aim in this subsection to establish the relationships between these stochastic orderings for Markov processes. First we study the case for the comparison of CTMCs, and secondly the monotonicity property.

**2.5.1. Strong and weak comparisons**

First, we can remark that the $\preceq_{st}$ ordering between the processes does not imply $\preceq_{wk}$ and $\preceq_{wk^*}$ orderings. If the $\preceq_{st}$ ordering exists between the two processes: \{X_{1}(n), n \geq 0\} and \{X_{2}(n), n \geq 0\}, then from definition 6, we have:
\[
p \preceq_{wk} q \implies pP_{1} \preceq_{wk} qP_{2}
\]

If \(p \preceq_{wk} q\), then we have two cases:

1. \(p \preceq_{st} q\), then from equation 13 we have: \(pP_{1} \preceq_{st} qP_{2}\), which implies that \(pP_{1} \preceq_{wk} qP_{2}\) due to the increasing sets family inclusions (see property 1).

2. If \(p \not\preceq_{st} q\) then we could have \(pP_{1} \not\preceq_{st} qP_{2}\), so we could not deduce that \(pP_{1} \preceq_{wk} qP_{2}\).

Therefore if \(\{X_{1}(n), n \geq 0\} \preceq_{st} \{X_{2}(n), n \geq 0\}\), then we could not deduce that \(\{X_{1}(n), n \geq 0\} \preceq_{wk} \{X_{2}(n), n \geq 0\}\). We now give an example in order to show it clearly.

**Example:** We consider \(E = \{(0,0), (0,1), (1,0), (1,1)\}\) and the component-wise ordering on the state space. In the following infinitesimal generators the states are ordered in the lexicographic order.
\[
Q_{1} = \begin{pmatrix}
-0.5 & 0.25 & 0.25 & 0 \\
0 & -0.5 & 0.5 & 0 \\
0.25 & 0 & -0.5 & 0.25 \\
0 & 0.25 & 0.25 & -0.5
\end{pmatrix}
\]

- \(\Phi_{wk}(E) = \{(0,0), (0,1), (1,0), (1,1)\}\)
- \(\Phi_{st}(E) = \Phi_{wk}(E) \cup (E - \{(0,0)\})\), where
\[
\{-0,0\} = E
- \{(0,1)\} = \{(0,1), (1,1)\}
- \{(1,0)\} = \{(1,0), (1,1)\}
- \{(1,1)\} = \{(1,1)\}
-E - \{(0,0)\} = \{(0,1), (1,0), (1,1)\}.
\]

Let take \(\lambda = |Q_{1}| + |Q_{2}| = 1\) which is both greater than \(|Q_{1}|\) and \(|Q_{2}|\).
\[
P_{1} = I + Q_{1} = \begin{pmatrix}
0.5 & 0.25 & 0.25 & 0 \\
0.25 & 0.5 & 0.25 & 0 \\
0.25 & 0.25 & 0.5 & 0
\end{pmatrix}
\]
\[
P_{2} = I + Q_{2} = \begin{pmatrix}
0.5 & 0.25 & 0.25 & 0 \\
0.25 & 0.5 & 0.25 & 0 \\
0.25 & 0.25 & 0.5 & 0
\end{pmatrix}
\]

We can observe that \(\forall \Gamma \in \Phi_{st}(E)\):
\[
\sum_{z \in \Gamma} P_{1}(x, z) \leq \sum_{z \in \Gamma} P_{2}(y, z), \forall x \preceq y
\]

so we deduce that:
\[
\{X_{1}(t), t \geq 0\} \preceq_{st} \{X_{2}(t), t \geq 0\}
\]

We now show that:
\[
\{X_{1}(t), t \geq 0\} \not\preceq_{wk} \{X_{2}(t), t \geq 0\}
\]

We use definition 6. Let consider two probability vectors \(p\) and \(q\) such that \(p = (0.5, 0.5, 0.5, 0)\), and \(q = (0.5, 0, 0, 0.5)\). It follows from definition 3 that \(p \not\preceq_{wk} q\). However for the increasing set \(\Gamma_{1} = \{(0,1), (1,0), (1,1)\} \in \Phi_{st}(E)\), the sense of inequality is reversed as \(\sum_{x \in \Gamma_{1}} p[x] = 1 > \sum_{x \in \Gamma_{1}} q[x] = 0.5\). Thus \(p \preceq_{wk} q\) and \(p \not\preceq_{st} q\). By multiplying with matrices:
\[
pP_{1} = (0.125, 0.25, 0.5, 0.125)
qP_{2} = (0.25, 0.25, 0.25, 0.25)
\]

If we take the increasing set \(\Gamma = \{(1,0), (1,1)\} \in \Phi_{wk}(E)\), we have
\[
\sum_{x \in \Gamma} pP_{1}[x] = 0.625 \text{ and } \sum_{x \in \Gamma} qP_{2}[x] = 0.5
\]

We deduce that \(pP_{1} \not\preceq_{wk} qP_{2}\). This result is important because if we have the $\preceq_{st}$ ordering between Markov...
processes, then we could not deduce that weaker orderings as \( \preceq_{wk} \) exist. In the case when the \( \preceq_{st} \) -ordering cannot be established between the processes, then one must verify if weaker orderings exist or not by applying increasing set method, (theorem 5).

### 2.5.2. Strong and weak monotonicity

In this subsection, we show that the \( \preceq_{st} \) monotonicity does not imply the \( \preceq_{wk} \) monotonicity. From definition 8, the \( \preceq_{st} \) monotonicity is defined as

\[
p \preceq_{st} q \implies pP \preceq_{st} q P
\]

If \( p \preceq_{wk} q \), and \( p \preceq_{st} q \) then by applying equation 14, we could not conclude on a relation between \( pP \) and \( qP \). We give the following example about these remarks.

**Example:** Let \( \{X(t), t \geq 0\} \) be a CTMC with the following infinitesimal generator \( Q \):

\[
Q = \begin{pmatrix}
-0.5 & 0.25 & 0.25 & 0 \\
0 & -0.5 & 0.5 & 0 \\
0.25 & 0 & -0.5 & 0.25 \\
0 & 0.25 & 0.25 & -0.5
\end{pmatrix}
\]

we define \( \{X^{\lambda}(n), n \geq 0\} \) the uniformised Markov chain where \( \lambda = 1 = 2|Q| \), with probability transition matrix:

\[
P^{\lambda} = I + Q
\]

We can easily remark that \( \{X^{\lambda}(n), n \geq 0\} \) is \( \preceq_{st} \)-monotone as

\[
\forall x \preceq y \in E, \sum_{z \in \Gamma} P^{\lambda}(x, z) \leq \sum_{z \in \Gamma} P^{\lambda}(y, z), \forall \Gamma \in \Phi_{st}(E)
\]

However \( \{X(n), n \geq 0\} \) is not \( \preceq_{wk} \) monotone. Take \( p = (0.0, 0.5, 0.5, 0) \) and \( q = (0.5, 0, 0, 0.5) \), we see that \( p \preceq_{wk} q \). We obtain \( pP^{\lambda} = (0.125, 0.25, 0.25, 0.125) \), and \( qP^{\lambda} = (0.25, 0.25, 0.25, 0.25) \). For \( \Gamma = \{(1, 0), (1, 1)\} \), we have

\[
\sum_{x \in \Gamma} pP^{\lambda}[x] = 0.625 > \sum_{x \in \Gamma} qP^{\lambda}[x] = 0.5
\]

Thus \( \{X^{\lambda}(n), n \geq 0\} \) is not \( \preceq_{wk} \) monotone, and we deduce that \( \{X(t), t \geq 0\} \) is not \( \preceq_{wk} \) monotone.

These results are important in order to know which method to choose to establish a stochastic ordering between Markov processes. Although the coupling of sample path generates the \( \preceq_{st} \) ordering, it is not the case for \( \preceq_{wk} \) and \( \preceq_{wk^*} \) orderings. The same remark can be deduced for the monotonicity. Next, we explain how to apply the stochastic comparison methods for performance evaluation studies.

### 2.6. Performance measure bounds

It is not trivial to choose \( \preceq_{\Phi} \in \{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}\} \) since each stochastic ordering generates different comparisons between underlying probability distributions. More precisely, one has to see which kind of increasing set corresponds to the subset \( A \) over which the performance measure of interest is specified (see equation 1). Thus \( A \) is the subset on which we must check the probability constraints.

From previous sections, we can explain how to apply stochastic comparison methods in order to compute performance measure bounds. First, \( A \) must be an increasing set \((A \in \Phi_{st}(E))\) to be able to establish inequalities. Moreover it follows from Property 1 that, if \( A \in \Phi_{wk}(E) \) then it is also \( A \in \Phi_{wk^*}(E) \).

1. If \( A \in \Phi_{wk}(E) \), then we apply the coupling method in order to check if the \( \preceq_{st} \) ordering exists. So we have two cases:

   - If the \( \preceq_{st} \) ordering exists, then for the computation of transient probability distributions bounds, we have to take the initial probability vectors \( p \) and \( q \) such that \( p \preceq_{st} q \) in definition 5 whatever the increasing set \( A \). As an example, if \( A \in \Phi_{wk^*}(E) \), then we could not take \( p \preceq_{st} q \) as \( \preceq_{st} \)-comparisons do not imply \( \preceq_{wk} \) comparisons.

   - If the \( \preceq_{st} \) ordering does not exist, then we look for weaker orderings. For example, if \( A \in \Phi_{wk}(E) \) then we check if the \( \preceq_{wk} \) ordering exists by applying theorem 5. We check if the \( \preceq_{wk} \)-monotonicity exists (not the \( \preceq_{st} \)-monotonicity because it does not imply the \( \preceq_{wk} \)-monotonicity).

2. If \( A \notin \Phi_{st}(E) \), then in the case of an upper bound (resp. a lower bound) we define an increasing set \( B \) such that \( A \subset B \) (resp. \( B \subset A \)), and we return to (1) by considering \( B \) instead of \( A \).

For instance, if we consider a queueing network formalism and component-wise order on the state space, loss probabilities can be derived from \( \preceq_{wk} \) (tail distributions) while resource utilisation can be derived from \( \preceq_{wk^*} \) comparisons. The \( \preceq_{st} \) ordering imposes more constraints, which may degrade the accuracy of bounds. But it allows us to build bounds for more performance measures, as both loss probabilities and resource utilisation. For instance, in [4], we have defined different bounding systems from \( \preceq_{st} \) and \( \preceq_{wk} \) orderings. As the loss probability bounds can be derived from these two kinds of stochastic orderings, we have compared the quality of the bounding systems with respect to this measure.
3. QUEUEING SYSTEM ANALYSIS

The system understudy is a general network defined as a graph with a set nodes (representing routers, switches, servers) and edges interconnecting the nodes (representing the transmission medium). We propose to model this system by a queueing network similar to a Jackson network except that queues have finite capacities. The system is represented by \( n \) queues, and each queue \( i \) has a finite capacity \( K_i \), and is characterised by the following parameters:

- Exponential inter-arrival times, with parameters \( \lambda_i \);
- Exponential service times, with parameters \( \mu_i \).

After a service, a customer transits from queue \( i \) to queue \( j \) with probability \( p_{ij} \) if the queue is not full, otherwise it is lost. A customer leaves the network with probability \( d_i \), the customer goes out. We assume that \( p_{ii} = 0 \), and \( \sum_{j \neq i} p_{ij} + d_i = 1 \), \( \forall i = 1 \ldots n \).

Let \( \{X(t), t \geq 0\} \) be the Markov process representing the evolution of this system with infinitesimal generator \( Q \). We denote by \( I \) the stationary probability distribution which has no product form solution. Thus its computation is very difficult due to the state space explosion with \( n \) the number of queues. We propose to use the stochastic comparisons in order to define bounding processes easier to analyse. We study two different ways for the definition of these systems. In order to easily compute the stationary probability distribution of \( \{X(t), t \geq 0\} \) we propose to make the queues independent in order to obtain a product form. Two kinds of systems are defined: the first one is obtained by removing the links between the queues, and so the bounding system is represented by independent \( M/M/1/K_i \) queues. For both systems, the stationary probability distribution can be computed easily, as they have a product form.

We apply stochastic comparisons in order to prove that these systems really provide bounds. On a multidimensional and partially ordered state space, different stochastic orderings could be defined as strong and weak orderings. The stochastic comparison for the strong ordering called \( \preceq_{st} \) -comparison used in this paper is based on the coupling of the processes. It remains to compare the realisations using the events occurring in the systems. We apply the \( \preceq_{st} \) -comparison to prove that the Jackson Network represents an upper bound for the system understudy. The strong ordering between processes could be very useful in performance evaluation as it generates the comparison of performance measures written as increasing functions on the (stationary and transient) distributions. So we can compare performance measures such as blocking probabilities, delays, and resource utilisation. As the strong ordering requires hard constraints, in some cases it could not be defined between the processes. So it could be interesting to check if weaker orderings could be defined. As the strong ordering could not exist between the system understudy and the system represented by the \( n \) independent \( M/M/1/K_i \) queues, then we propose to apply the increasing sets method \( [11, 10] \) to prove that the \( \preceq_{wk} \) ordering exists. We call \( \preceq_{wk} \)-comparison the stochastic comparison based on increasing sets in order to generate the \( \preceq_{wk} \)-ordering. Note that in \([12]\), other bounding systems have been defined by generalising the approach to any partition of the set of nodes.

Some interesting features will be studied in these stochastic comparison methods. For the \( \preceq_{st} \)-comparison, we compare a Markov process defined on a finite state space with another on an infinite state space. For the \( \preceq_{wk} \)-comparison, we define the increasing sets from events, in order to limit the number of increasing sets effectively used for the comparison. We compute the blocking probability as the performance measure from bounding models and compare them under different input parameters. The processes understudy are multidimensional, defined on \( E = \mathbb{N}^n \). We propose to use the component-wise partial ordering denoted by \( \preceq \) on this state space:

\[
\forall x, y \in \mathbb{N}^n, x \preceq y \iff x_i \leq y_i, \forall i = 1, \ldots, n \quad (15)
\]

In the next section, we define a bounding system with infinite queue capacities. Thus we have a Jackson network having product form solution for the stationary distribution. Although, it is intuitive that making capacities infinite provides upper bounds on the number of customers, we need to formally prove by considering all events.

3.1. The coupling with Jackson network

We propose to bound the process \( \{X(t), t \geq 0\} \) with a another process with the same assumptions, except the queues have infinite capacities, so the bounding system represents a Jackson network. The stationary probability distribution can be derived easily, using the product form. We denote by \( \{S(t), t \geq 0\} \) the bounding process. Using the coupling, we aim to prove the following proposition:

**Proposition 1**

\[
\{X(t), t \geq 0\} \preceq_{st} \{S(t), t \geq 0\} \quad (16)
\]

**Proof:** In most of the cases, the coupling concerns processes which have either both infinite
Let suppose that \( \hat{\Gamma} \) has a finite state space, or both finite state space. Here, it is not the case as \( \{X(t), t \geq 0\} \) has a finite state space, and \( \{S(t), t \geq 0\} \) is an infinite. From the coupling method applied for stochastic comparisons, we prove that there exist two processes \( \{X(t), t \geq 0\} \) (resp. \( \{S(t), t \geq 0\} \)) with the same infinitesimal generator matrix than \( \{X(t), t \geq 0\} \) (resp. \( \{S(t), t \geq 0\} \)) representing two different realisations and we prove that:

\[
\tilde{X}(0) \preceq \tilde{S}(0) \Rightarrow \tilde{X}(t) \preceq \tilde{S}(t), \quad t > 0 \tag{17}
\]

Remember that \( \{X(t), t \geq 0\} \) is a multidimensional process on \( E \), it is represented by the vector:

\[
X(t) = (X_1(t), \ldots, X_i(t), \ldots, X_n(t)) \tag{18}
\]

also for \( \{\tilde{X}(t), t \geq 0\} \) and \( \{\tilde{S}(t), t \geq 0\} \).

Let suppose that \( \tilde{X}(t) \preceq \tilde{S}(t) \). We show if \( \tilde{X}(t + \Delta t) \preceq \tilde{S}(t + \Delta t) \) by considering the evolution from events occurring during the time interval \( \Delta t \):

1. an arrival in queue \( i \): the arrival rate in queue \( i \) is \( \lambda_i \) from \( \tilde{X}(t) \) (if queue \( i \) is not full) and also from \( \tilde{S}(t) \). So if \( \tilde{X}(t) \) increases with an arrival in queue \( i \), then \( \tilde{S}(t) \) will increase also. From the component \( \tilde{X}_i(t) \), we obtain \( \tilde{X}_i(t + \Delta t) = \min\{K_i, \tilde{X}_i(t) + 1\} \), and from \( \tilde{S}_i(t) \) as the capacity is infinite, the component always increases: \( \tilde{S}_i(t + \Delta t) = \tilde{S}_i(t) + 1 \). Since other components do not change, and \( \tilde{X}(t) \preceq \tilde{S}(t) \) then \( \tilde{X}(t + \Delta t) \preceq \tilde{S}(t + \Delta t) \).

2. a transit from queue \( i \) to queue \( j \): as the transition rate is \( \mu_{ij} p_{ji} \) for \( \tilde{X}(t) \) and \( \tilde{S}(t) \), the evolutions are the same. The transit occurs if \( \tilde{X}_i(t) > 0 \) and the customer is accepted in queue \( j \) if \( \tilde{X}_j(t) < K_j \), otherwise it is lost. From \( \tilde{X}(t) \), we obtain \( \tilde{X}_i(t + \Delta t) = \max\{0, \tilde{X}_i(t) - 1\} \), and \( \tilde{X}_j(t + \Delta t) = \min\{K_j, \tilde{X}_j(t) + 1\} \). From \( \tilde{S}(t) \), similarly, \( \tilde{S}_i(t + \Delta t) = \max\{0, \tilde{S}_i(t) - 1\} \), and as the queue \( j \) is infinite, \( \tilde{S}_j(t + \Delta t) = \tilde{S}_j(t) + 1 \). Since other components do not change, and \( \tilde{X}(t) \preceq \tilde{S}(t) \) then \( \tilde{X}(t + \Delta t) \preceq \tilde{S}(t + \Delta t) \).

3. a service from queue \( i \) to the outside: as the service rate is \( \nu_i d_i \) for the two processes, if we have a service in queue \( i \) for \( \tilde{S}(t) \), we have also a service for \( \tilde{S}(t) \). So \( \tilde{X}_i(t + \Delta t) = \max\{0, \tilde{X}_i(t) - 1\} \), and \( \tilde{S}_i(t + \Delta t) = \max\{0, \tilde{S}_i(t) - 1\} \). Then \( \tilde{X}(t + \Delta t) \preceq \tilde{S}(t + \Delta t) \).

Since the process \( \{S(t), t \geq 0\} \) is time-homogeneous, even if it is defined on an infinite state space, the coupling of the processes for the comparison of the realisations is still verified. So we deduce that \( \{X(t), t \geq 0\} \preceq_{st} \{S(t), t \geq 0\} \), and so \( \{S(t), t \geq 0\} \) represents a strong bounding system.

We have the comparison of transient probability distributions: \( P(X(t) \in \Gamma) \leq P(S(t) \in \Gamma), \quad \forall \Gamma \in \Phi_{st}(E) \). If the stability condition is satisfied, then the stationary probability distribution \( \Pi_S \) exists. So we have the following inequality:

\[
\sum_{x \in \Gamma} \Pi(x) \leq \sum_{x \in \Gamma} \Pi^S(x), \quad \forall \Gamma \in \Phi_{st}(E) \tag{19}
\]

Next we try to define another bounding system by modifying the interconnections between the queues.

### 3.2. Weak comparisons with independent \( M/M/1/K_i \) queues

We define a bounding process represented by \( n \) independent \( M/M/1/K_i \) queues, obtained from the exact system by removing the links between the queues [11, 10]. For each pairs of queues \( j \) and \( i \), the flow of packets leaving node \( i \) and entering node \( j \) with rate \( \mu_{ij} p_{ji} \) is forbidden. As compensation this flow is added to the flow of packets entering node \( i \). So each queue \( i \) is an \( M/M/1/K_i \) with an arrival rate \( \lambda_i + \sum_{j \neq i} \mu_{ij} p_{ji} \), and a service rate \( \mu_i \). The interest of this bounding system is that both stationary and transient behavior are known. We denote by \( W(t) \) the Markov process representing the evolution of this system, with infinitesimal generator \( Q_W \). We denote by \( \Pi_W \) the stationary probability distribution. As it has been presented in [10], we could not have the \( \preceq_{st} \)-ordering between \( \{X(t), t \geq 0\} \) and \( \{W(t), t \geq 0\} \). Next, we explain how to compare the processes using weak ordering \( \preceq_{wk} \).

We will explain in this section how to compare \( \{X(t), t \geq 0\} \) and \( \{W(t), t \geq 0\} \) using the increasing set formalism. We give the following theorem:

**Proposition 2**

\[
\{X(t), t \geq 0\} \preceq_{wk} \{W(t), t \geq 0\} \tag{20}
\]

**Proof:** In [10] similar systems with infinite queues have been studied. The \( \preceq_{wk} \)-comparison has been proved using an operator-analytic approach. In this paper, we define increasing sets using a formalism based on events in order to provide a more intuitive approach for the \( \preceq_{wk} \)-comparison. We apply theorem 5 for the \( \preceq_{wk} \) ordering. The increasing set theory is not easy to apply because the stochastic comparison is performed on all the increasing sets of the considered family. Therefore for multidimensional state spaces, as the state space increases exponentially, the number of increasing sets will be also very large. We propose to solve this problem by defining only the increasing sets which
are necessary for the comparison. There are two steps in theorem 5: first we must verify the monotonicity of one of the processes, and secondly we have to compare the transition rates of the processes in the increasing sets. Since \( W(t) \) is the product of birth and death processes, it is \( \preceq_{wk} \)-monotone [11].

Next, we verify the condition (2) of theorem 5, so we compare the transition rates of each process using the increasing sets: we have to verify if \( \sum_{x \in \Gamma} Q(x, z) \) is lower than \( \sum_{x \in \Gamma} Q^W(x, z) \), \( \forall \Gamma \in \Phi_{wk}(E) \). As \( E \) is multidimensional, \( \Phi_{wk}(E) \) could be very large. We need to define the increasing sets which are necessary for the verification of the \( \preceq_{wk} \)-comparison. Since the transitions from any state happen due to events, we define the increasing sets from these events. Let \( e_i \) be a vector from \( N^m \) such that all components are null except component \( i \) which equals 1. From \( x \), we have three kinds of events:

- an arrival in queue \( i \): which generates a transition from \( x \) to \( x + e_i \). So we define the increasing set \( \{ x + e_i \} \).
- a service in queue \( i \): which generates a transition from \( x \) to \( x - e_i \). So we define the increasing set \( \{ x - e_i \} \).
- a transit from queue \( j \) to queue \( i \): which generates a transition from \( x \) to \( x - e_j + e_i \). So we define the increasing set \( \{ x - e_j + e_i \} \).

We add also the increasing set \( \{ x \} \) corresponding to the process staying in state \( x \). We denote by \( C_{wk}(E) \subset \Phi_{wk}(E) \) the set of increasing sets which are necessary for the \( \preceq_{wk} \)-comparison. So

\[
C_{wk}(E) = \{ \Gamma_{x+e_i}, \Gamma_{x-e_j+e_i}, \Gamma_{x-e_i} \}
\]

It is easy to see that

\[
\sum_{z \in \Gamma} Q(x, z) \leq \sum_{z \in \Gamma} Q^W(x, z), \forall \Gamma \in C_{wk}(E)
\]

Thus we can deduce from theorem 5 that \( \{ X(t), t \geq 0 \} \preceq_{wk} \{ W(t), t \geq 0 \} \), and we call \( \{ W(t), t \geq 0 \} \) a weak bounding system.

We denote by \( \Pi^W \) the stationary probability distribution of \( \{ W(t), t \geq 0 \} \). From the stochastic comparisons of the processes we have: \( P(X(t) \in \Gamma) \leq P(W(t) \in \Gamma), \forall \Gamma \in \Phi_{wk}(E) \). And so for the stationary probability distributions we have:

\[
\sum_{x \in \Gamma} \Pi(x) = \sum_{x \in \Gamma} \Pi^W(x), \forall \Gamma \in \Phi_{wk}(E)
\]

As we will see after, these results could be very interesting because the right term can be easily computed as the product of probability distributions of independent \( M/M/1/K_i \) queues.

### 4. Bounds on Blocking Probabilities

The goal of this section is to compute the blocking probabilities in order to study the accuracy of bounding systems. As bounding systems are generated either by the strong ordering or the weak ordering, the objective is also to conclude on which order could provide the most precise bound. First, we give blocking probability equations for the exact and bounding systems.

#### 4.1. Computation of blocking probabilities

The exact blocking probability \( Pb_i \) on queue \( i \) for the process \( \{ X(t), t \geq 0 \} \) is given by the following formula:

\[
Pb_i = \sum_{x \geq x^*} \Pi(x)
\]

where \( x^* \) is the vector where all components are null except the component \( i \) which equals \( K_i \). We can remark that \( Pb_i \) is very difficult to compute because there is not a product form for \( \Pi \) and the state space size is very large, equals to: \( K_1 \times \cdots \times K_i \times \cdots \times K_n \). So we propose to compute different blocking probability bounds for queue \( i \): the weak bound \( Pb_{iw}^i \) on the weak bounding system, and the strong bound \( Pb_{is}^i \) on the strong bounding system. First, we can remark that the set of states \( \Gamma = \{ x \geq x^* \} \) used for the computation of the blocking probability \( Pb_i \) is an increasing set such that \( \Gamma \in \Phi_{wk}(E) \), and also \( \Gamma \in \Phi_{st}(E) \) (see property 1). Secondly, from inequality 21, with \( \Gamma = \{ x \geq x^* \} \), for any queue \( i \) we have: \( Pb_i \leq Pb_{iw}^i \). Furthermore from inequality 19, we have also for any queue \( i \) : \( Pb_i \leq Pb_{is}^i \). We now explain how to compute the blocking probabilities \( Pb_{iw}^i \) and \( Pb_{is}^i \). The blocking probability \( Pb_{iw}^i \) is equivalent to the blocking probability in an \( M/M/1/K_i \) queue:

\[
Pb_{iw}^i = a_i K_i \frac{1 - a_i}{1 - a_i + \gamma}, \text{ with } a_i = \frac{\lambda_i}{\mu_i}
\]

The blocking probability \( Pb_{is}^i \) is computed from the process \( S(t) \) as follows: \( Pb_{is}^i = \sum_{x \geq x^*} \Pi^S(x) \). As \( \{ S(t), t \geq 0 \} \) represents the Jackson network, then the stationary probability distribution \( \Pi^S \) is computed as the product of the stationary probability distributions of each queue. Let \( \Lambda_i \) be the input traffic in queue \( i \) : it equals the sum of traffic coming from the outside \( \lambda_i \) and the traffic coming from other queues \( k (k \neq i) : \Lambda_k p_{ki} \). We denote by:

\[
b_i = \frac{\Lambda_i}{\mu_i} \text{ where } \Lambda_i = \lambda_i + \sum_{k=1, k \neq i}^n \Lambda_k p_{ki}
\]
We suppose that the stability condition $b_i < 1$ is satisfied, so the stationary probability distribution could be computed. The blocking probability of the upper bound $S(t)$ for queue $i$ denoted $Pb_i^s$ can be computed easily:

$$Pb_i^s = \sum_{x_i=K_i}^{\infty} b_i^{x_i} (1 - b_i)$$  \hspace{1cm} (25)

Since $\sum_{x_i=0}^{\infty} b_i^{x_i} (1 - b_i) = 1$, then $Pb_i^s = b_i^{K_i}$.

For the comparison of the blocking probabilities $Pb_i^s$ and $Pb_i^w$, then first, we compare $a_i$ with $b_i$. As $\Lambda_k < \mu_k, \forall 1 \leq k \leq n$, then:

$$\lambda_i + \sum_{k=1, k\neq i}^{n} \Lambda_k p_{ki} \leq \lambda_i + \sum_{k=1, k\neq i}^{n} \mu_k p_{ki}$$  \hspace{1cm} (26)

thus we deduce that $b_i \leq a_i$, and also $b_i^{K_i} \leq a_i^{K_i}$. But as: $\frac{1-a_i}{1-b_i} < 1$, then we could not conclude for the comparison between $Pb_i^s$ and $Pb_i^w$. In the next section compare these values in a queueing network with respect to different parameter values: (capacities, arrival and service rates).

### 4.2. Numerical Results

We give now some numerical results in order to study the quality of the bounds. As we need to compare the different bounds with the same input parameters, we choose them under the stability conditions of the strong bound: $b_i < 1, \forall 1 \leq i \leq n$. First, we study a simple system in order to compare the exact blocking probabilities with bounding measures. The system is represented by two queues: queue 1 and queue 2 in tandem. The arrival rate in queue 1 is $\lambda_1=100$, and the service rate $\mu_1=110$, the probability $p_{12}=1$, and $d_1=0$. For queue 2, we have $\lambda_2=0$, and $\mu_2=110$. So $a_2=0.95$, and $b_2=0.90$. In Table 1, we give the exact blocking probabilities $Pb_2$ of queue 2 (obtained from QNAP simulator) and also upper bounding measures $Pb_2^w$ and $Pb_2^s$, according to the buffer size $K_2$. We can easily see that $Pb_2^w$ provides better bounds for $K_2 = 20, 30$, and for upper values of $K_2$, $Pb_2^s$ is better. Next, we study

![Figure 1: Network understudy](image)

a more complex network given in Fig. 1 with 8 nodes (routers), and edges corresponding to transmission medium. We propose to study the dimensioning problem in order to guarantee a blocking probability threshold. We define a queueing model for the quantitative analysis of this system. We suppose that the packet processing time at each node due to routing treatments is negligible. So the waiting time at nodes is due only to the packet transmission delay depending on the transmission rate of the link. So each link interconnecting two nodes is modelled by a queue. The input parameters for each queue are given in Table 2. For each queue $i$, we give the external packet arrival rate $\lambda_i$, the service rate $\mu_i$, and the non-null routing probabilities $p_{ij}$ to queues $1 \leq j \leq n$ interconnected to $i$. The customers go out of the network with probability $d_i$. To simplify the study, we suppose that all packets have the same length. The goal of our study is to compute blocking probabilities of queue 9. As the exact blocking probability $Pb_9$ is very difficult to compute due to

<table>
<thead>
<tr>
<th>$K_2$</th>
<th>$Pb_2$ (Exact)</th>
<th>$Pb_2^w$ (Weak)</th>
<th>$Pb_2^s$ (Strong)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>$0.137 \times 10^{-4}$</td>
<td>$0.28 \times 10^{-4}$</td>
<td>$0.148$</td>
</tr>
<tr>
<td>30</td>
<td>$0.485 \times 10^{-2}$</td>
<td>$0.14 \times 10^{-4}$</td>
<td>$0.57 \times 10^{-1}$</td>
</tr>
<tr>
<td>40</td>
<td>$0.178 \times 10^{-2}$</td>
<td>$0.83 \times 10^{-2}$</td>
<td>$0.22 \times 10^{-2}$</td>
</tr>
<tr>
<td>50</td>
<td>$0.643 \times 10^{-9}$</td>
<td>$0.48 \times 10^{-2}$</td>
<td>$0.8 \times 10^{-2}$</td>
</tr>
<tr>
<td>60</td>
<td>$0.223 \times 10^{-9}$</td>
<td>$0.29 \times 10^{-2}$</td>
<td>$0.32 \times 10^{-2}$</td>
</tr>
<tr>
<td>70</td>
<td>$0.14 \times 10^{-3}$</td>
<td>$0.18 \times 10^{-2}$</td>
<td>$0.12 \times 10^{-2}$</td>
</tr>
<tr>
<td>80</td>
<td>$0.268 \times 10^{-3}$</td>
<td>$0.11 \times 10^{-2}$</td>
<td>$4.88 \times 10^{-4}$</td>
</tr>
<tr>
<td>90</td>
<td>$0.199 \times 10^{-6}$</td>
<td>$0.700 \times 10^{-3}$</td>
<td>$1.88 \times 10^{-4}$</td>
</tr>
<tr>
<td>100</td>
<td>$0.787 \times 10^{-9}$</td>
<td>$4.377 \times 10^{-4}$</td>
<td>$7.2565 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

**Table 1: Blocking probabilities**

<table>
<thead>
<tr>
<th>Queue : i</th>
<th>link</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>nodes 1 and 2</td>
</tr>
<tr>
<td>2</td>
<td>nodes 2 and 3</td>
</tr>
<tr>
<td>3</td>
<td>nodes 2 and 6</td>
</tr>
<tr>
<td>4</td>
<td>nodes 3 and 4</td>
</tr>
<tr>
<td>5</td>
<td>nodes 3 and 5</td>
</tr>
<tr>
<td>6</td>
<td>nodes 6 and 7</td>
</tr>
<tr>
<td>7</td>
<td>nodes 4 and 7</td>
</tr>
<tr>
<td>8</td>
<td>nodes 5 and 7</td>
</tr>
<tr>
<td>9</td>
<td>nodes 7 and 8</td>
</tr>
</tbody>
</table>

**Table 2: Model queues definition**

<table>
<thead>
<tr>
<th>Queue : i</th>
<th>$\lambda_i$</th>
<th>$\mu_i$</th>
<th>routing probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>520</td>
<td>600</td>
<td>$p_{12} = 0.5$, $p_{13} = 0.5$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>400</td>
<td>$p_{24} = 0.5$, $p_{25} = 0.5$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>300</td>
<td>$p_{36} = 1$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>200</td>
<td>$p_{47} = 1$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>200</td>
<td>$p_{58} = 1$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>270</td>
<td>$p_{69} = 1$</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>150</td>
<td>$p_{79} = 1$</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>150</td>
<td>$p_{89} = 1$</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>600</td>
<td>$d_9 = 1$</td>
</tr>
</tbody>
</table>

**Table 3: Input parameter values**
We choose $\mu$ decrease in Table 4 we can see that the weak bound is better when the buffer size is large, the strong bound is provides better results. At the end, we can conclude and $b$ as in table 3, except $b_9$ = 0.95, $b_1$ = 0.86. We take the same input parameters values (for queues interconnected to queue 9 i.e. queues 6, 7, and 8), so $a_9$ = 0.98, $b_9$ = 0.896, and the blocking probability bounds are given in Table 5. We can also remark that the weak bound provides better results only for small buffer sizes. When the buffer size increases, the strong bound is better. We aim to increase the load of queues interconnected to queue 9. We suppose that external arrival rates of these queues are: $\lambda_6 = 5$, $\lambda_7 = 15$, $\lambda_8 = 15$. Others parameters are the same than in table 3. We obtain $b_9$ = 0.925, and $a_9$ = 0.95. We can see easily in Table 6, that the weak bound provides always better results than the strong bound. In this case, we have $\Lambda$ which is very close to $\mu_9$ (for queues interconnected to queue 9 i.e. queues 6, 7, and 8), so $a_9$ is very close to $b_9$. Furthermore, as the blocking probability of the weak bound is computed from a finite capacity system, it provides better results. We propose now to decrease significantly the load of queue 9. We take the same input parameters values as in table 3, except $\mu_9$ = 800. We obtain $a_9$ = 0.712, and $b_9$ = 0.65, and the blocking probabilities are given in Table 7. We can see that the strong bound provides better results. At the end, we can conclude that when the buffer size is large, the strong bound is better, and when the load is high the weak bound is the best. So we can remark easily that if $b_9$ is high (0.9), the strong bound gives worse results than the weak bound. Intuitively when the load is high, the weak bound is close to the exact system. Moreover, the weak bound is generated from a finite capacity system (as the exact system). On the other hand, when $b_9$ decreases, the strong bound could provide better results especially when the buffer size increases.

In this section we have tried to discuss the cases where the weak bound (or the strong bound) is better, according to input parameters. As we have seen, it is difficult to identify the best bounds according to the input parameter values. So the solution is to compute the two upper bounds, and to choose the minimum between them in order to have a precise idea about the exact blocking probability. Hence, for a threshold $t$, if $\min(P_{b_9}^w, P_{b_9}^s) < t$, then we are sure that $P_{b_9} < t$. We now explain how to use these results for the resource dimensioning.

### 4.3. Buffer dimensioning

We study in this system the dimensioning problem of queue 9, using blocking probability bounds. The goal of this study is to give the buffer size which provides a blocking probability lower than a threshold $t$. So we look for the value of $K_9$ such that $P_{b_9} < t$. As $P_{b_9}$ is very difficult to obtain, we compute two bounds $P_{b_9}^s$ and $P_{b_9}^w$ such that $P_{b_9} \leq P_{b_9}^s$, and

<table>
<thead>
<tr>
<th>$K_9$</th>
<th>$P_{b_9}^w$ (Weak)</th>
<th>$P_{b_9}^s$ (Strong)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.027</td>
<td>0.057</td>
</tr>
<tr>
<td>30</td>
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<td>0.0136</td>
</tr>
<tr>
<td>40</td>
<td>0.0073</td>
<td>0.0032</td>
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<tr>
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<td>0.0024</td>
<td>1.866 $\times 10^{-4}$</td>
</tr>
<tr>
<td>70</td>
<td>0.00141</td>
<td>4.4632 $\times 10^{-5}$</td>
</tr>
<tr>
<td>80</td>
<td>8.389 $\times 10^{-4}$</td>
<td>1.067 $\times 10^{-5}$</td>
</tr>
<tr>
<td>90</td>
<td>4.9910 $\times 10^{-4}$</td>
<td>2.55 $\times 10^{-6}$</td>
</tr>
<tr>
<td>100</td>
<td>2.97 $\times 10^{-4}$</td>
<td>6.098 $\times 10^{-7}$</td>
</tr>
</tbody>
</table>

### Table 4: Blocking probability bounds: $a_9 = 0.95$, $b_9 = 0.86$

<table>
<thead>
<tr>
<th>$K_9$</th>
<th>$P_{b_9}^w$ (Weak)</th>
<th>$P_{b_9}^s$ (Strong)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.039</td>
<td>0.11</td>
</tr>
<tr>
<td>30</td>
<td>0.0284</td>
<td>0.0377</td>
</tr>
<tr>
<td>40</td>
<td>0.0168</td>
<td>0.0126</td>
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<tr>
<td>50</td>
<td>0.0122</td>
<td>0.0042</td>
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<tr>
<td>60</td>
<td>0.0092</td>
<td>0.00142</td>
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<tr>
<td>70</td>
<td>0.0071</td>
<td>1.7893 $\times 10^{-4}$</td>
</tr>
<tr>
<td>80</td>
<td>0.0056</td>
<td>1.607 $\times 10^{-4}$</td>
</tr>
<tr>
<td>90</td>
<td>0.00453</td>
<td>5.3923 $\times 10^{-5}$</td>
</tr>
<tr>
<td>100</td>
<td>0.0036</td>
<td>1.809410 $\times 10^{-5}$</td>
</tr>
</tbody>
</table>

### Table 5: Blocking probability bounds: $b_9 = 0.896$, $a_9 = 0.98$

<table>
<thead>
<tr>
<th>$K_9$</th>
<th>$P_{b_9}^w$ (Weak)</th>
<th>$P_{b_9}^s$ (Strong)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>3.271 $\times 10^{-4}$</td>
<td>1.812 $\times 10^{-4}$</td>
</tr>
<tr>
<td>30</td>
<td>1.102 $\times 10^{-5}$</td>
<td>2.44 $\times 10^{-6}$</td>
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<tr>
<td>40</td>
<td>3.715 $\times 10^{-7}$</td>
<td>3.2849 $\times 10^{-8}$</td>
</tr>
<tr>
<td>50</td>
<td>1.252 $\times 10^{-8}$</td>
<td>4.422 $\times 10^{-10}$</td>
</tr>
<tr>
<td>60</td>
<td>4.242 $\times 10^{-10}$</td>
<td>5.953 $\times 10^{-12}$</td>
</tr>
<tr>
<td>70</td>
<td>1.42410 $\times 10^{-11}$</td>
<td>8.015 $\times 10^{-14}$</td>
</tr>
<tr>
<td>80</td>
<td>4.80 $\times 10^{-13}$</td>
<td>1.079 $\times 10^{-15}$</td>
</tr>
<tr>
<td>90</td>
<td>1.619 $\times 10^{-14}$</td>
<td>1.452 $\times 10^{-17}$</td>
</tr>
<tr>
<td>100</td>
<td>5.450 $\times 10^{-16}$</td>
<td>1.955 $\times 10^{-19}$</td>
</tr>
</tbody>
</table>

### Table 6: Blocking probability bounds: $a_9 = 0.95$, $b_9 = 0.925$

<table>
<thead>
<tr>
<th>$K_9$</th>
<th>$P_{b_9}^w$ (Weak)</th>
<th>$P_{b_9}^s$ (Strong)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.02718</td>
<td>0.21</td>
</tr>
<tr>
<td>30</td>
<td>0.0134</td>
<td>0.0964</td>
</tr>
<tr>
<td>40</td>
<td>0.0073</td>
<td>0.0442</td>
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<tr>
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<td>0.0041</td>
<td>0.0202</td>
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<tr>
<td>60</td>
<td>0.0024</td>
<td>0.0093</td>
</tr>
<tr>
<td>70</td>
<td>0.0014</td>
<td>0.00426</td>
</tr>
<tr>
<td>80</td>
<td>8.389 $\times 10^{-4}$</td>
<td>0.0019</td>
</tr>
<tr>
<td>90</td>
<td>4.9910 $\times 10^{-4}$</td>
<td>8.969 $\times 10^{-4}$</td>
</tr>
<tr>
<td>100</td>
<td>2.97 $\times 10^{-4}$</td>
<td>4.113 $\times 10^{-4}$</td>
</tr>
</tbody>
</table>

### Table 7: Blocking probability bounds $a_9 = 0.712$, $b_9 = 0.65$
Strong and Weak Stochastic Comparisons

<table>
<thead>
<tr>
<th>t</th>
<th>Weak bound</th>
<th>Strong bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-4}</td>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td>10^{-10}</td>
<td>70</td>
<td>60</td>
</tr>
<tr>
<td>10^{-15}</td>
<td>100</td>
<td>90</td>
</tr>
</tbody>
</table>

Table 8: Buffer size dimensioning for $a_9 = 0.712, b_9 = 0.65$

<table>
<thead>
<tr>
<th>t</th>
<th>Weak bound</th>
<th>Strong bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-1}</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>10^{-2}</td>
<td>40</td>
<td>60</td>
</tr>
<tr>
<td>10^{-3}</td>
<td>80</td>
<td>90</td>
</tr>
</tbody>
</table>

Table 9: Buffer size dimensioning for $a_9 = 0.95, b_9 = 0.925$

If $Pb_0 \leq Pb_0$, if $Pb_0^w < t$ or $Pb_0^s < t$, then we are sure that $Pb_0 < t$. As sometimes the weak bound is lower (or upper) than the strong bound, we compute the two bounds and we choose the lower buffer size $K_0$ obtained by the bounds. From results obtained in table 7, we give the different buffer sizes obtained by weak and strong bounds for different threshold $t$ of the blocking probability. The buffer dimensioning for different values of $t$ is given in the table 8. We can see that the Strong bound gives always the lowest buffer size. Furthermore, in the case of blocking probabilities given in table 6, the weak bound gives better results. So the buffer dimensioning deduced from table 6 shows clearly that the weak bound gives the lowest buffer size (see table 9).

5. CONCLUSION

In this paper we propose to define different bounding systems by using two simplification concepts: cutting links or making infinite the queues. We prove using stochastic comparisons that strong or weak orderings could be defined, and we propose to study the accuracy of bounds. We compute blocking probability bounds, and we compare the values derived from the bounding systems for different input parameters. In order to dimension buffers in a network, one can use the most accurate bound among the proposed ones. As a future work, it will be interesting to propose other bounding systems by using the two concepts of simplification at the same time, in order to improve the quality of the bounds. It will be interesting to define bounding systems such that some queues are making independent and others have infinite capacities, according to the input parameters.

REFERENCES